ON THE EXISTENCE PROBLEM OF THE TOTAL DOMINATION VERTEX CRITICAL GRAPHS

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Abstract. The existence problem of the total domination vertex critical graphs has been studied in a series of articles. The aim of the present article is twofold. First, we settle the existence problem with respect to the parities of the total domination number m and the maximum degree Δ : for even m except $m = 4$, there is no m-γ_t-critical graph regardless of the parity of Δ; for $m = 4$ or odd $m \geq 3$ and for even Δ, an m-γ_t-critical graph exists if and only if $\Delta \geq 2\lfloor \frac{m-1}{2} \rfloor$; for $m = 4$ or odd $m \geq 3$ and for odd Δ, if $\Delta \geq 2\lfloor \frac{m-1}{2} \rfloor + 7$, then m-γ_t-critical graphs exist, if $\Delta < 2\lfloor \frac{m-1}{2} \rfloor$, then m-γ_t-critical graphs do not exist. The only remaining open cases are $\Delta = 2\lfloor \frac{m-1}{2} \rfloor + k$, $k = 1, 3, 5$. Second, we study these remaining open cases when $m = 4$ or odd $m \geq 9$. As the previously known result for $m = 3$ [1, 2], we also show that for $\Delta(G) = 3, 5, 7$, there is no 4-γ_t-critical graph of order $\Delta(G) + 4$. On the contrary, it is shown that for odd $m \geq 9$ there exists an m-γ_t-critical graph for all $\Delta \geq m - 1$.

1. Introduction

A domination and its variations in graph theory have been studied widely and extensively because of its rich applications [2, 6, 8, 11]. Two books by Haynes, Hedetniemi and Slater provide a well written survey on this subject [4, 5]. We refer to [4] for notation and general terminology.

Let $G = (V(G), E(G))$ be a simple graph of order $n(G)$. The minimum degree and the maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A subset $S \subseteq V$ is *a dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of dominating sets. A subset $S \subseteq V$ is *a total dominating set* of G if every vertex of G is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of total dominating sets. A total dominating set of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-*set*.

Goddard et al. introduced the concept of total domination critical graphs [2]. A graph G with no isolated vertex is *total domination vertex critical* if for any vertex v of G that is not adjacent to a leaf, a vertex of degree one, the total domination number of $G - v$ is less than the total domination number of G. Such a graph is said to be γ_t-*critical* or m-γ_t-*critical* if its total domination number is m. It is well known that the order of m-γ_t-critical graph G is at least $\Delta(G) + m$. So, they suggested the following classification problem of the total domination critical graphs.

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Problem 1 ([2]). *Characterize m-γ_t-critical graphs G with order $\Delta(G)+m$.*

There have been a series of articles regarding this problem. Mojdeh and Rad found 3-γ_t-critical graphs of order $3+\Delta(G)$ for any even $\Delta(G)$ and showed that there is no 3-γ_t-critical graph G of order $3+\Delta(G)$ for $\Delta(G)=3,5$ [11]. In [1], Chen and Sohn proved that there is no 3-γ_t-critical graph of order $\Delta(G)+3$ with $\Delta(G)=7$ and $\delta(G)\geq 2$. Furthermore, they gave a family of 3-γ_t-critical graphs of order $\Delta(G)+3$ with odd $\Delta(G)\geq 9$ and $\delta(G)\geq 2$. Hassankhani and Rad proved that there is no 4-γ_t-critical graph of order $\Delta(G)+4$ with $\delta(G)\geq 2$ for $\Delta(G)=3,5$ [3]. There have been several partial results on the existence problem of the total domination vertex critical graphs from different point of views.

The aim of the present article is twofold. First, we settle the existence problem with respect to the parities of the total domination number m and the maximum degree Δ in Theorem 2.

Theorem 2. *If there exists an m-γ_t-critical graph of order $\Delta+m$ for some Δ then $m=4$ or $m\geq 3$ is odd. Conversely, for any $m=4$ or odd $m\geq 3$,*

(1) *if $\Delta<2\lfloor\frac{m-1}{2}\rfloor$, then there exists no m-γ_t-critical graph of order $\Delta+m$.*
(2) *For any even $\Delta\geq 2\lfloor\frac{m-1}{2}\rfloor$, there exists an m-γ_t-critical graph of order $\Delta+m$.*
(3) *For any odd $\Delta\geq 2\lfloor\frac{m-1}{2}\rfloor+7$, there exists an m-γ_t-critical graphs of order $\Delta+m$.*

Theorem 2 implies that the only remaining cases are $\Delta = 2\lfloor\frac{m-1}{2}\rfloor+k$, $k=1,3,5$. Second, we study these remaining open cases when $m=4$ or odd $m\geq 9$. When $m=4$, we show that there is a 4-γ_t-critical graph G of order $\Delta(G)+4$ with $\delta(G)\geq 2$ if and only if $\Delta(G)=2,4,6,8$ or $\Delta(G)\geq 9$. For odd $m\geq 9$, it is shown that there exists an m-γ_t-critical graph G of order $\Delta(G)+m$ with $\delta(G)\geq 2$ if and only if $\Delta(G)\geq m-1$.

The outline of this paper is as follows. In section 2, we review some definitions and previous results. In section 1, some properties of m-γ_t-critical graph of order $\Delta+m$ will be given. In section 4, we provide the proof of the Theorem 2. In section 5, we deal with the remaining open cases for $m=4$ and $m\geq 9$.

2. Preliminaries

In this section, we review some definitions and previous results. The degree, neighborhood and closed neighborhood of a vertex v in a graph G are denoted by $d(v)$, $N(v)$ and $N[v]=N(v)\cup\{v\}$, respectively. For a subset S of V, we set $N(S)=\bigcup_{v\in S}N(v)$ and $N[S]=N(S)\cup S$. The graph induced by $S\subseteq V$ is denoted by $G[S]$. The cycle, path and complete graph on n vertices are denoted by C_n, P_n and K_n, respectively. A vertex of degree one is called a *leaf*. A vertex v of G is called a *support vertex* if it is adjacent to a leaf. Let $S(G)$ be the set of all support vertices of G. The *corona* of a graph H, denoted by $cor(H)$, is the graph obtained from H by adding a leaf adjacent to each vertex of H.

For two graphs G_1 and G_2 and for two vertices $v_1\in V(G_1)$ and $v_2\in V(G_2)$, a *vertex amalgamation* of G_1 and G_2 with two vertices v_1 and v_2 is a graph whose vertex set is $(V(G_1)-v_1)\cup(V(G_2)-v_2)\cup\{v\}$ and edge set is

$$E(G_1-v_1)\cup E(G_2-v_2)\cup\{vu|v_1u\in E(G_1)\}\cup\{vw|v_2w\in E(G_2)\}.$$

The vertex amalgamation method is useful to construct a new γ_t-critical graph by the following proposition.

Proposition 3 ([2]). *Let F and H be j-γ_t-critical and k-γ_t-critical graphs, respectively, with minimum degrees at least two and let G be a graph formed by identifying a vertex of F with a vertex of H. If $\gamma_t(G) = j + k - 1$ then G is also γ_t-critical.*

Lemma 4. *For any $i = 1, 2$, let G_i be an m_i-γ_t-critical graph G_i of order $\Delta(G_i) + m_i$ with $\delta(G_i) \geq 2$ and let $v_i \in V(G_i)$ be a vertex of maximum degree in G_i. If each component of $G[V(G_i) - N[v_i]]$ is a P_2 then the vertex amalgamation G of G_1 and G_2 with v_1 and v_2 is an $(m_1 + m_2 - 1)$-γ_t-critical graph of order $\Delta(G) + m_1 + m_2 - 1$, where $\Delta(G) = \Delta(G_1) + \Delta(G_2)$.*

Proof. Let v be the vertex of G whose degree is $\Delta(G) = \Delta(G_1) + \Delta(G_2)$, namely, v is an amalgamated vertex. For any $u \in N(v)$, $(V(G) - N[v]) \cup \{u\}$ is a total dominating set of G and whose cardinality is $m_1 + m_2 - 1$. Hence $\gamma_t(G) \leq m_1 + m_2 - 1$. Let S be a $\gamma_t(G)$-set of G. Suppose $v \in S$. Then, v is adjacent to a vertex $u \in S - \{v\}$. Without loss of generality, we may assume that $u \in V(G_1)$. Then, $(V(G_1) \cap (S - \{v\})) \cup \{v_1\})$ is a total dominating set of G_1. Furthermore, for S to dominate $G_2 - N[v]$, we have $|V(G_2) \cap (S - \{v\})| \geq m_2 - 1$. Hence, $|S| \geq m_1 + m_2 - 1$, which means that $\gamma_t(G) = m_1 + m_2 - 1$. By Proposition 3, G is an $(m_1 + m_2 - 1)$-γ_t-critical graph of order $\Delta(G) + m_1 + m_2 - 1$. □

The following two lemmas are known results in [2] which will be used in this paper.

Lemma 5 ([2]). *If G is a γ_t-critical graph, then $\gamma_t(G - v) = \gamma_t(G) - 1$ for every $v \in V - S(G)$. Furthermore, a $\gamma_t(G - v)$-set contains no neighbor of v.*

Lemma 6 ([2]). *If a graph G has nonadjacent vertices u and v such that $v \notin S(G)$ and $N(u) \subseteq N(v)$, then G is not γ_t-critical.*

Mojdeh and Rad [11] found the following lemma about a total domination vertex critical graph G of order $\Delta(G) + \gamma_t(G)$ with $\delta(G) \geq 2$.

Lemma 7 ([11]). *There is no 3-γ_t-critical graph G of order $\Delta(G) + 3$ with $\Delta(G) = 3$, 5 and $\delta(G) \geq 2$.*

3. Some properties of γ_t-critical graph G with $\gamma_t(G) = n - \Delta(G)$

In this section, we find some properties of γ_t-critical graph G with $\gamma_t(G) = n - \Delta(G)$. Throughout the section, we assume the following notation for γ_t-critical graph G with $\gamma_t(G) = n - \Delta(G)$ and $\delta(G) \geq 2$ unless stated otherwise. Let v be a vertex whose degree is the maximum degree $\Delta(G)$. Since G is γ_t-critical, it follows that $\gamma_t(G - v) = \gamma_t(G) - 1 = n - \Delta(G) - 1$. Let S be a γ_t-set of $G - v$. Then, $S = V(G) - N[v]$ by Lemma 5. Let $H_1, H_2, \cdots, H_t$ be the components of $G[S]$ and let $V(H_i) = S_i$ for all $i = 1, 2, \ldots, t$. We find the following two lemmas regarding the γ_t-critical graph G with $\gamma_t(G) = n - \Delta(G)$ and $\delta(G) \geq 2$.

Lemma 8. *Every γ_t-critical graph G with $\gamma_t(G) = n - \Delta(G)$ and $\delta(G) \geq 2$ is connected.*

Proof. Suppose that G is not connected. Then, at least one of H_1, H_2, $\cdots$, H_t is also a connected component of G, say H_i is such a component. Since $\delta(G) \geq 2$, $|V(H_i)| = |S_i| \geq 3$. Choose a spanning tree T of H_i and one end vertex u of T. Then, $S_i - u$ is a total dominating set of H_i and furthermore $S - u$ is a total dominating set of $G - v$, which is a contradiction. □

Lemma 9. *Let G be a γ_t-critical graph with $\gamma_t(G) = n - \Delta(G)$ and $\delta(G) \geq 2$. Then,*

(1) *H_i is a P_2 or a P_3 for $i = 1, 2, \cdots, t$.*
(2) *If $G[S]$ contains a P_3 component, then $G[S] = P_3$. Furthermore, for the $P_3 = u_1u_2u_3$, $N(u_2) \cap N(v) = \emptyset$ and $N(v)$ is a disjoint union of nonempty sets $N(u_1) - u_2$ and $N(u_3) - u_2$.*
(3) *If H_i is a P_2 for all $i = 1, 2, \ldots, t$, i.e., $H_i = u_iw_i$, then for any $u \in S$, $N(u) \cap N(v) \neq \emptyset$ and $N(v)$ is a disjoint union of $N(u_1) - w_1, N(w_1) - u_1, \ldots, N(u_t) - w_t, N(w_t) - u_t$.*

Proof. (1) First, we aim to show that $\Delta(H_i) \leq 2$ for $i = 1, 2, \cdots, t$. Suppose that $\Delta(H_j) \geq 3$ for some $1 \leq j \leq t$. Let u be a vertex of H_j whose degree in H_j is at least 3. Choose a spanning tree T of H_j containing all edges incident to u. Then, T has at least three leaves. Let u_1, u_2, u_3 be three leaves in T. For any $x \in N(v)$, let $S' = (S - \{u_2, u_3\}) \cup \{v, x\}$. Then, S' is a total dominating set of G and hence $\gamma_t(G) \leq |S'| = |S| = \gamma_t(G) - 1$, which is a contradiction. Therefore, $\Delta(H_i) \leq 2$ for $i = 1, 2, \cdots, t$. It implies that H_i is a path or a cycle for $i = 1, 2, \cdots, t$.

Suppose that there exists j such that H_j is a cycle $u_1u_2 \cdots u_ku_1$ for $k \geq 3$. Then, there is u_ℓ such that $N(u_\ell) \cap N(v) \neq \emptyset$. Without loss of generality, we assume $N(u_1) \cap N(v) \neq \emptyset$ and pick a vertex $x \in N(u_1) \cap N(v)$. Then, $S'' = (S - \{u_2, u_3\}) \cup \{v, x\}$ is a total dominating set of G, which is a contradiction. So, for all $i = 1, 2, \ldots, t$, H_i is a path.

Suppose that there exists a path $H_i = u_1u_2 \cdots u_k$ for $k \geq 4$. Then, $(S - \{u_1, u_k\}) \cup \{v, x\}$ for some $x \in N(v)$ is a total dominating set of G, which is a contradiction. Therefore, H_i is a P_2 or a P_3 for all $i = 1, 2, \cdots, t$.

(2) Let $G[S]$ contains a P_3 component, say $u_1u_2u_3$. If $G[S]$ contains another component $w_1w_2w_3$ which is isomorphic to P_3, then for some $x \in N(v)$, $(S - \{u_3, w_3\}) \cup \{v, x\}$ is a total dominating set of G, which is a contradiction. Next if we suppose $G[S]$ contains a P_3 and at least one P_2, say w_1w_2. Then, $N(v) \cap N(w_1) \neq \emptyset$ because $\delta(G) \geq 2$. For some $x \in N(v) \cap N(w_1)$, $(S - \{u_3, w_2\}) \cup \{v, x\}$ is a total dominating set of G, it leads us a contradiction. Therefore, $G[S] = P_3 = u_1u_2u_3$.

Since $\delta(G) \geq 2$, $(N(u_i) - u_2) \cap N(v) \neq \emptyset$ for any $i = 1$ or 3. If $N(u_2) \cap N(v) \neq \emptyset$ then for any $x \in N(u_2) \cap N(v)$, $\{v, x, u_2\}$ is a total dominating set of G, which is a contradiction. Hence, $N(v)$ is a disjoint union of nonempty sets $N(u_1) - u_2$ and $N(u_3) - u_2$.

(3) Since $\delta(G) \geq 2$, $N(u) \cap N(v) \neq \emptyset$ for any $u \in S$. Furthermore, for any $x \in N(v)$, $N(x) \cap S \neq \emptyset$ because S is a total dominating set of $G - v$. We want to show that $|N(x) \cap S| = 1$ for any $x \in N(v)$. Suppose that there exists an $x \in N(v)$ such that $u_i, w_i \in N(x)$ for some $i = 1, 2, \ldots, t$. Then, $S' = (S - \{u_i, w_i\}) \cup \{v, x\}$ is a total dominating set of G, which is a contradiction.

For the next case, suppose that there exists an $x \in N(v)$ such that $u_i, u_j \in N(x)$ for some different i, j. Choose $y_i \in N(v) \cap N(w_i)$ and $y_j \in N(v) \cap N(w_j)$. Then, one can easily check that $(S - \{u_i, u_j, w_i, w_j\}) \cup \{v, x, y_i, y_j\}$ is a total dominating set of G, which is a contradiction. Similarly, one can show that a contradiction occurs if $|N(x) \cap S| \geq 2$ for some $x \in N(v)$. It implies that $N(v)$ is a disjoint union of $N(u_1) - w_1, N(w_1) - u_1, \ldots, N(u_t) - w_t, N(w_t) - u_t$. □

These results can be summarized to obtain general figures of γ_t-critical graph G with $\gamma_t(G) = n - \Delta(G)$ and $\delta(G) \geq 2$ as in Figure 1.

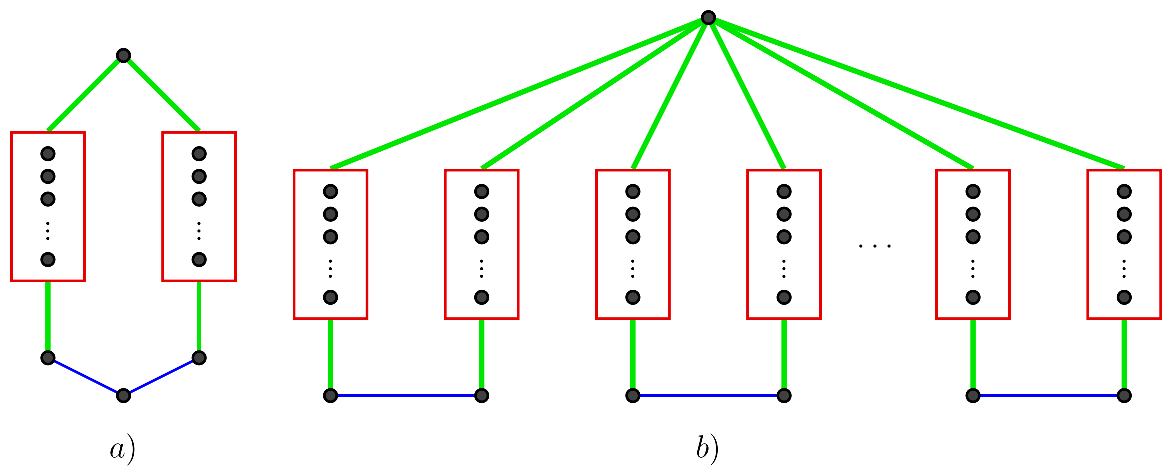

FIGURE 1. Figures of γ_t-critical graph G with $\gamma_t(G) = n - \Delta(G)$ and $\delta(G) \geq 2$ where all vertices in the boxes are adjacent to vertices connected to boxes by thick lines and there could be edges between vertices in different boxes or in the same box. This convention will be used for other figures.

4. Proof of Theorem 2

In this section, we shall give a proof of Theorem 2. Suppose that G is an m-γ_t-critical graph of order $\Delta(G) + m$. Let v be a vertex for which $d(v) = \Delta(G)$. By Lemma 9, each connected component of $G[V(G) - N[v]]$ is a P_2 or a P_3 and if there exists a component P_3 then $G[V(G) - N[v]] = P_3$. Hence, $m - 1 = |G[V(G) - N[v]]|$ is 3 or even. It implies that $m = 4$ or $m \geq 3$ is odd.

If $m = 4$, then $G[V(G) - N[v]]$ is a P_3 and $\Delta(G) \geq 2$ by Lemma 9 (2). If $m \geq 3$ is odd then each component of $G[V(G) - N[v]]$ is a P_2 and $\Delta(G) \geq m - 1$ by Lemma 9 (3). Hence, for $m = 4$ or odd $m \geq 3$ if $\Delta < 2\lfloor \frac{m-1}{2} \rfloor$, then there exists no m-γ_t-critical graph of order $\Delta + m$.

For $m = 4$ and for even $\Delta \geq 2$, let G be a graph whose vertex set is $\{v\} \cup (U \cup W) \cup \{u_1, u_2, u_3\}$ with $|U| = |W| = \Delta/2$ and whose edge set is composed of $\{vx, vy, u_1x, u_3y | x \in U,\ y \in W\} \cup \{u_1u_2, u_2u_3\}$ as in Figure 1 a) and the subgraph induced by the vertices in between U and W is $K_{\Delta/2,\Delta/2} - E(M)$, where M is an 1-factor of $K_{\Delta/2,\Delta/2}$. Then, one can show that G is a 4-γ_t-critical graph of order $\Delta(G) + 4$.

For odd $m \geq 3$ and for even $\Delta \geq m - 1$, let G_1 be a graph whose vertex set is $\{v_1\} \cup (U_1 \cup W_1) \cup \{u_1, w_1\}$ with $|U_1| = |W_1| = (\Delta - m + 3)/2$ and whose edge set is composed of $\{v_1x, v_1y, u_1x, w_1y | x \in U_1,\ y \in W_1\} \cup \{u_1w_1\}$ and the subgraph induced by the vertices in between U_1 and W_1 is $K_{(\Delta-m+3)/2,(\Delta-m+3)/2} - E(M)$, where M is an 1-factor of $K_{(\Delta-m+3)/2,(\Delta-m+3)/2}$. Then, one can show that G_1 is a 3-γ_t-critical graph of order $\Delta(G_1) + 3 = \Delta - m + 6$. Note that C_5 is a 3-γ_t-critical graph of order 5. So, the vertex amalgamation G of G_1 and $(m-3)/2$ 5-cycles with v_1 and any vertex in each $(m-3)/2$ 5-cycles as in Figure 2 is an m-γ_t-critical graph of order $\Delta - m + 6 + 4 \cdot \frac{m-3}{2} = \Delta + m$ by Proposition 4. Hence, for $m = 4$ or odd $m \geq 3$ and for any even $\Delta \geq 2\lfloor \frac{m-1}{2} \rfloor$, there exists an m-γ_t-critical graph of order $\Delta(G) + m$.

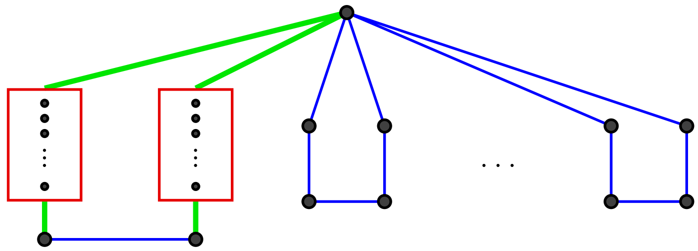

FIGURE 2. Figures of m-γ_t-critical graph of order $\Delta + m$, where each box contains $\frac{\Delta-m+3}{2}$ vertices and the subgraph induced by the vertices in two boxes is $K_{(\Delta-m+3)/2,(\Delta-m+3)/2} - E(M)$, where M is a 1-factor of $K_{(\Delta-m+3)/2,(\Delta-m+3)/2}$.

Now, we want to consider odd Δ. In the paper [11], Mojdeh and Rad showed that there is no 3-γ_t-critical graph G of order $\Delta(G) + 3$ for $\Delta(G) = 3, 5$. In [1], Chen and Sohn proved that there is no 3-γ_t-critical graph of order $\Delta(G) + 3$ with $\Delta(G) = 7$. Furthermore, they gave a family of 3-γ_t-critical graphs of order $\Delta(G) + 3$ with $\Delta(G) \geq 9$. For any odd $m \geq 3$ and for any odd $\Delta \geq m + 6$, let G_2 be a 3-γ_t-critical graph of order 12 with $\Delta(G_2) = 9$ and $\delta(G_2) \geq 2$ and let G_3 be an $(m-2)$-γ_t-critical graph of order $\Delta + m - 11$ with $\Delta(G_3) = \Delta - 9 \geq m - 3$ and $\delta(G_3) \geq 2$. Let $v_i \in V(G_i)$ be a vertex such that $d(v_i) = \Delta(G_i)$ for each $i = 2, 3$. Then, the vertex amalgamation G of G_2 and G_3 with the vertices v_2 and v_3 is an m-γ_t-critical graph of order $\Delta + m$ with $\Delta(G) = \Delta$ and $\delta(G_3) \geq 2$ by Proposition 4. In the next section, we construct a 4-γ_t-critical graph of order $\Delta + 4$ for any odd $\Delta \geq 9$. Hence, for any $m = 4$ or odd $m \geq 3$ and for any odd $\Delta \geq 2\lfloor \frac{m-1}{2} \rfloor + 7$, there exists an m-γ_t-critical graph of order $\Delta + m$.

5. $m = 4$ OR ODD $m \geq 9$

The only remaining open cases are $\Delta = 2\lceil \frac{m-1}{3} \rceil + k$, $k = 1, 3, 5$. In this section, we prove that there is no 4-γ_t-critical graph of order $\Delta + 4$ with $\delta(G) \geq 2$ for $\Delta = 3, 5$ or 7. For odd $m \geq 9$, it will be shown that there exists an m-γ_t-critical graph of order $\Delta + m$ with $\delta(G) \geq 2$ for any odd $\Delta \geq 2\lceil \frac{m-1}{3} \rceil + 1$.

Theorem 10. *There is no 4-γ_t-critical graph G of order $\Delta(G) + 4$ with $\Delta(G) = 3, 5, 7$ and $\delta(G) \geq 2$.*

Proof. Let G be a γ_t-critical graph with $\gamma_t = n - \Delta(G)$ and $\delta(G) \geq 2$. For any vertex $u \in V(G)$, let S_u be a $\gamma_t(G-u)$-set. Choose $v \in V(G)$ such that $d(v) = \Delta(G)$. Since $n(G) = \Delta(G) + 4$, we can assume that $V(G) - N[v] = \{u, z, w\}$. Since G is 4-γ_t-critical, by Lemma 5, it follows that $S_v = \{u, z, w\}$ and $N(u) \cup N(w) - \{z\} = N(v)$. Furthermore, $N(u) \cap N(w) = \{z\}$. Otherwise, say $x \in N(u) \cap N(w)$, then $\{v, x, u\}$ is a $\gamma_t(G)$-set, which is a contradiction.

Suppose that $|N(u) \cap N(v)| \geq 2$ and $|N(w) \cap N(v)| \geq 2$. Then, for any $x \in N(u) \cap N(v)$, $S_x = \{z, w, y\}$ or $\{w, y, x_1\}$ for some $y \in N(w) \cap N(v)$ and $x_1 \in N(u) \cap N(v)$. If $S_x = \{z, w, y\}$ then y dominates all elements in $N(u) \cap N(v) - \{x\}$ and hence, $\{w, y, x_2\}$ is also a total dominating set of $G - x$ for any $x_2 \in N(u) \cap N(v) - \{x\}$. Therefore, we assume that for any $t \in N(v)$, $|S_t \cap N(v)| \geq 2$ in the case $|N(u) \cap N(v)| \geq 2$ and $|N(w) \cap N(v)| \geq 2$.

It divides into three cases depending on $\Delta(G)$.
Case 1. $\Delta(G) = 3$. We assume that $N(u) \cap N(v) = \{x_1\}$ and $N(w) \cap N(v) = \{y_1, y_2\}$. Since $G - y_2$ is the cycle C_6 which has a total domination number 4. It is a contradiction.
Case 2. $\Delta(G) = 5$. It divides into two cases depending on $|N(u) \cap N(v)|$.
Case 2.1. We assume that $N(u) \cap N(v) = \{x_1\}$ and $N(w) \cap N(v) = \{y_1, y_2, y_3, y_4\}$. It is obvious that there is no edges x_1y_j $(j = 1, 2, 3, 4)$ in G. If we delete y_1, there is the cycle C_6 in G which have a total domination number 4. It is a contradiction.
Case 2.2. We assume that $N(u) \cap N(v) = \{x_1, x_2\}$ and $N(w) \cap N(v) = \{y_1, y_2, y_3\}$. It is obvious that for any $i = 1, 2, 3$, y_i cannot be adjacent to both x_1 and x_2. Without loss of generality, we can assume that $x_1y_1, x_1y_2 \notin E(G)$. It implies that $S_{x_2} = \{x_1, y_3, w\}$, $x_1y_3 \in E(G)$ and $x_2y_3 \notin E(G)$. By considering S_{y_3}, one can show that $x_2y_1 \in E(G)$ or $x_2y_2 \in E(G)$. Let $x_2y_1 \in E(G)$. Then, $S_{y_1} = \{x_1, y_3, u\}$ and $y_2y_3 \in E(G)$. Furthermore, $S_{y_2} = \{x_2, y_1, u\}$ and $y_1y_3 \in E(G)$. In this case, $\{x, y_3, u\}$ is a total dominating set of G, a contradiction.
Case 3. $\Delta(G) = 7$. It divides into three cases depending on $|N(u) \cap N(v)|$.
Case 3.1. We assume that $N(u) \cap N(v) = \{x_1\}$ and $N(w) \cap N(v) = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. It is obvious that G is not 4-γ_t-critical graph.
Case 3.2. We assume that $N(u) \cap N(v) = \{x_1, x_2\}$ and $N(w) \cap N(v) = \{y_1, y_2, y_3, y_4, y_5\}$. By the Pigeonhole Principle, we can assume that $x_1 \in S_{y_1} \cap S_{y_2} \cap S_{y_3}$. For $j = 1, 2, 3$, $S_{y_j} \cap \{y_4, y_5\} \neq \emptyset$. By the Pigeonhole Principle, we can assume that $S_{y_1} = S_{y_2} = \{x_1, y_4, u\}$. Since $\{x_1, y_4, u\}$ is a $\gamma_t(G - y_1)$-set and $x_1y_2 \notin E(G)$, $y_2y_4 \in E(G)$. Therefore $\{x_1, y_4, u\}$ is not a $\gamma_t(G - y_2)$-set. It is a contradiction.
Case 3.3. We assume that $N(u) \cap N(v) = \{x_1, x_2, x_3\}$ and $N(w) \cap N(v) = \{y_1, y_2, y_3, y_4\}$. It divides into four cases depending on existing edges between $\{x_1, x_2, x_3\}$. Suppose that there is no edges in $\{x_1, x_2, x_3\}$. Without loss of generality, let $S_{x_1} = \{x_2, y_1, w\}$. Then, $x_2y_1, x_3y_1 \in E(G)$ and $x_1y_1 \notin E(G)$. By the similar way, we can assume that $x_1y_2, x_3y_2 \in E(G)$ and $x_1y_3, x_2y_3 \in E(G)$. Furthermore, $x_2y_2, x_3y_3 \notin E(G)$. Considering S_{y_4}, we may assume that $S_{y_4} = \{x_1, y_2, u\}$. Then, $y_1y_2 \in E(G)$ and $x_1y_4, y_2y_4 \notin E(G)$. If $x_2y_4 \in E(G)$ or $x_3y_4 \in E(G)$ then $\{x_2, y_1, u\}$ or $\{x_3, y_1, u\}$ is a $\gamma_t(G)$-set, which is a contradiction. Hence, we may assume that $x_2y_4, x_3y_4 \notin E(G)$. It implies that $S_{y_2} = \{x_2, y_3, u\}$ and hence $y_3y_4 \in E(G)$. Let us consider S_{y_3}. Since $y_2y_4 \notin E(G)$, $S_{y_3} = \{x_3, y_1, u\}$. It implies that $y_1y_4 \in E(G)$. Then, $\{x_2, y_1, u\}$ is a $\gamma_t(G)$-set, a contradiction.

If there is one edges in $\{x_1, x_2, x_3\}$, we assume that $x_2x_3 \in E(G)$. Without loss of generality, let $S_{x_1} = \{x_2, y_1, w\}$. Then, $x_2y_1 \in E(G)$ and $x_1y_1 \notin E(G)$. Also, without loss of generality, we may assume that $S_{x_2} = \{x_1, y_2, w\}$. It implies that $x_1y_2 \in E(G)$ and $x_3y_2 \in E(G)$. In this case, $\{x_3, y_2, w\}$ is a $\gamma_t(G)$-set. It is a contradiction.

If there is two or three edges in $\{x_1, x_2, x_3\}$, one can similarly get a contradiction as the case that there is one edge in $\{x_1, x_2, x_3\}$. $\square$

Lemma 11. *Let G be a connected graph with $\Delta(G) = 9$ or $\Delta(G) \geq 11$. Then there are positive integers $3, 2 = s_1, s_2 = s_3$ satisfying the following two conditions;*

(1) $3 + 2 + s_2 + s_3 = \Delta(G)$
(2) $2 = s_1 \leq s_2 = s_3$.

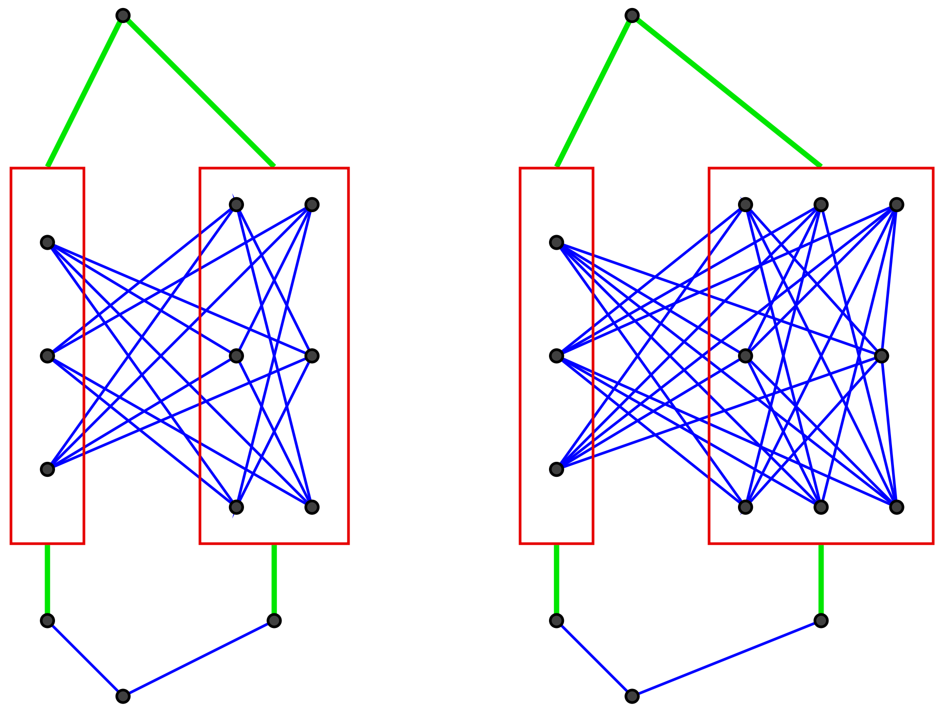

FIGURE 3. Figures of 4-γ_t-critical graphs with $\Delta(G) = 9, 11$.

Now we construct a family of 4-γ_t-critical graphs of order $\Delta(G) + 4$ with $\delta(G) \geq 2$ and $\Delta(G) = 9$ or $\Delta(G) \geq 11$.

Let H be a copy of the complement graph $\overline{K_3}$ of the complete graph K_3 . Let $V(H) = \{x_1, x_2, x_3\}$. Let H_i be a graph with a vertex set $V(H_i) = \{y_{i1}, y_{i2}, \cdots, y_{is_i}\}$ for $i = 1, 2, 3$. Suppose that $2 = s_1 \leq s_2 = s_3$. Let F be the graph obtained from $H_1 \cup H_2 \cup H_3$ by adding edges $y_{1j}y_{2k}$, $y_{2k}y_{3\ell}$, $y_{1j}y_{3\ell}$ for $j = 1, 2$, $k = 1, 2, \cdots, s_2$, and $\ell = 1, 2, \cdots, s_3$, $j \neq k$, $j \neq \ell$, $k \neq \ell$. Let G be the graph obtained from $H \cup F$ and four new vertices v, u, z, w by adding edges x_iy_{jk} for $1 \leq i, j \leq 3$, $i \neq j$ and $1 \leq k \leq s_j$, and then joining v to every vertex in $H \cup F$, joining u to every vertex in H and joining w to every vertex in F, and adding the edges uz and zw. Then $\Delta(G) = 3 + 2 + s_2 + s_3$. Two figures in Figure 3 are examples of 4-γ_t-critical graphs with $\Delta(G) = 9, 11$.

Theorem 12. *The graph G in Figure 3 is 4-γ_t-critical.*

Proof. It is obvious that $\gamma_t(G) = 4$. So we only prove that G is γ_t-critical graph. First, $\{v, y_{11}, w\}$, $\{v, x_1, u\}$, $\{v, x_1, y_{11}\}$ and $\{u, w, z\}$ is a total dominating set of $G - u$, $G - w$, $G - z$ and $G - v$ respectively. For any vertex $x_i \in V(G)$, $\{w, y_{i1}, z\}$ is a total dominating set of $G - x_i$. For any vertex $y_{jk} \in V(G)$, It is easy to choose a total dominating set of $G - y_{jk}$, In general, for any vertex $a \in V(G)$, $\gamma_t(G - a) = 3$. So G is a 4-γ_t-critical graph. □

By Theorems 2, 10 and 12, we have the following corollary.

Corollary 13. *There is a 4-γ_t-critical graph G of order $\Delta(G) + 4$ with $\delta(G) \geq 2$ if and only if $\Delta(G) = 2, 4, 6, 8$ or $\Delta(G) \geq 9$.*

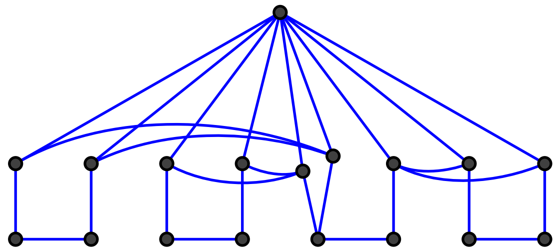

FIGURE 4. Figures of m γ_t-critical graph G with $\gamma_t(G) = n - \Delta(G)$ and $\delta(G) \geq 2$ for $m = 9$.

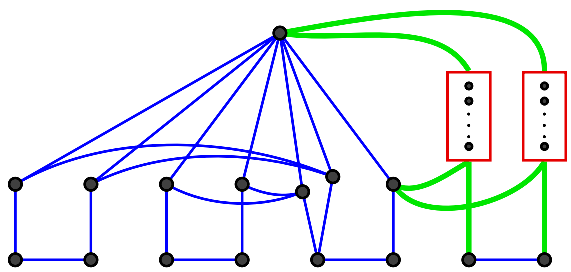

FIGURE 5. Figures of m γ_t-critical graph G with $\gamma_t(G) = n - \Delta(G)$ and $\delta(G) \geq 2$ for $m \geq 9$, where each box contains $\frac{\Delta-7}{2}$ vertices and the subgraph induced by the vertices in two boxes is $K_{\frac{\Delta-7}{2},\frac{\Delta-7}{2}} - E(M)$, where M is a 1-factor of $K_{\frac{\Delta-7}{2},\frac{\Delta-7}{2}}$.

From now on, we aim to consider an m-γ_t-critical graph G of order $\Delta(G) + m$ with $\delta(G) \geq 2$ for any odd $m \geq 9$ and odd $\Delta(G) \geq m$.

Theorem 14. *For any odd $m \geq 9$ and for any odd $\Delta \geq m$, there exists an m-γ_t-critical graph G of order $\Delta + m$ with $\Delta(G) = \Delta$ and $\delta(G) \geq 2$.*

Proof. Assume that there exists a 9-γ_t-critical graph G_1 of order $\Delta_1 + 9$ with $\Delta(G_1) = \Delta_1$ and $\delta(G_1) \geq 2$ for any odd $\Delta_1 \geq 9$. Then for odd $m \geq 9$ and for any odd $\Delta \geq m$, one can construct m-γ_t-critical graph G of order $\Delta + m$ with $\Delta(G) = \Delta$ and $\delta(G) \geq 2$ using a vertex amalgamation of G_1 and several C_5's. Hence, it suffices to show that there exists a 9-γ_t-critical graph G of order $\Delta + 9$ with $\Delta(G) = \Delta$ and $\delta(G) \geq 2$ for any odd $\Delta \geq 9$.

For any $\Delta \geq 9$, let $G = (V, E)$ be a graph whose vertex set is $\{v\} \cup \bigcup_{i=1}^{4} (U_i \cup W_i \cup \{u_i, w_i\})$, where

$$\begin{aligned} U_i &= \{x_i\} \text{ for } i = 1, 2,\ U_3 = \{x_{31}, x_{32}\}, U_4 = \{x_{41}, x_{42}, \ldots, x_{4\frac{\Delta-7}{2}}\}, \\ W_i &= \{y_i\} \text{ for } i = 1, 2, 3,\ W_4 = \{y_{41}, y_{42}, \ldots, y_{4\frac{\Delta-7}{2}}\} \end{aligned}$$

and its edge set is composed of

$$\begin{aligned} & \{vx, vy, xu_i, yw_i, u_iw_i \mid x \in U_i,\ y \in W_i,\ i = 1, 2, 3, 4\} \\ \cup\ & \{x_ix_{3i}, y_ix_{3i} \mid x_i \in U_i,\ y_i \in W_i,\ i = 1, 2\} \\ \cup\ & \{y_3x, y_3y \mid y_3 \in W_3,\ x \in U_4,\ y \in W_4\} \end{aligned}$$

as in Figure 4 and the subgraph induced by the vertices in U_4 and W_4 is $K_{\frac{\Delta-7}{2},\frac{\Delta-7}{2}} - E(M)$, where M is a 1-factor of $K_{\frac{\Delta-7}{2},\frac{\Delta-7}{2}}$. For our convenience, let $N_i = U_i \cup W_i \cup \{u_i, w_i\}$ for $i = 1,2,3,4$. We want to show that G is a 9-γ_t-critical graph of order $\Delta + 9$. Let S be a total dominating set of G. Then, one can check that $\gamma_t(G) = |S| \geq 8$ because for $i = 1,2,3,4$, $|S \cap N_i| \geq 2$ for S to dominate u_i and w_j. Suppose that $\gamma_t(G) = 8$. Then, $|S \cap N_i| = 2$ for any $i = 1,2,3,4$. Especially, $|S \cap N_3| = 2$. If $S \cap N_3 = \{x_{31}, u_3\}$ then for S to dominate y_3, $S \cap N_3$ is $\{x_{4j}, u_4\}$ or $\{y_{4j}, w_4\}$ for some $j = 1,2,\ldots,\frac{\Delta-7}{2}$. In either cases, W_4 or U_4 is not dominated. For other choices of $S \cap N_3$, one can similarly show that $V(G)$ is not totally dominated by S if $|S \cap N_3| = 2$. So, $\gamma_t(G) = |S| \geq 9$. For $S_1 = \{u_i, w_i \mid i = 1,2,4\} \cup \{v, x_{31}, u_3\}$, S_1 is total dominating set of G. Hence, $\gamma_t(G) = 9$.

If we delete u_j for some $j = 1,2,3,4$, then for some $y \in W_j$, $\{u_i, w_i \mid i = 1,2,3,4,\ i \neq j\} \cup \{v, y\}$ is a total dominating set of $G - u_j$. Hence, $\gamma_t(G - u_j) = 8$. Similarly, one can show that $\gamma_t(G - w_j) = 8$. If we delete x_1 from G then $\{u_i, w_i \mid i = 2,3,4\} \cup \{y_1, w_1\}$ is a total dominating set of $G - u_j$ and hence $\gamma_t(G - x_1) = 8$. If we delete $x_{3,1}$ from G then $\{u_1, w_1, x_2, y_2, x_{32}, y_3, x_{41}, y_{41}\}$ is a total dominating set of $G - x_{3,1}$ and hence $\gamma_t(G - x_{31}) = 8$. Similarly, one can show that for any $z \in V(G)$, $\gamma_t(G - z) = 8$. Therefore, G is a 9-γ_t-critical graph of order $\Delta + 9$. □

By Theorems 2 and 14, we have the following corollary.

Corollary 15. *For any odd $m \geq 9$, there exists an m-γ_t-critical graph G of order $\Delta(G) + m$ with $\delta(G) \geq 2$ if and only if $\Delta(G) \geq m - 1$.*

Remark: We settled the existence problem with respect to the parities of the total domination number m and the maximum degree Δ except some cases. The only remaining open cases are $\Delta = 5,7,9$ for $m = 5$ and $\Delta = 7,9,11$ for $m = 7$.

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